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1 维优化的锥模型方法的收敛阶①

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摘　要　基于锥模型的拟牛顿法已被许多研究者讨论过，并且 D. C. Sorensen 文(The Q-superlinear convergence of a collinear scaling algorithm for unconstrained optimization. SIAM J Numer Anal, 1980, 17(1):84～114)证明了该算法模型是超线性收敛的。本文中针对 1 维优化问题讨论了该算法模型的收敛阶，结果表明它是 Q-2阶收敛的，并且从极小点 x^* 的左右两边交错收敛到 x^*。

关键词　1 维优化；锥模型方法；收敛阶
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Convergence Rate of Conic Methods for One-dimensional Unconstrained Optimization

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Abstract　Conic methods for unconstrained optimization have been discussed by many people, and its superlinear convergence rate has been proved by D. C. Sorensen. In this paper, it is proved that the convergence rate order of conic method for one-dimensional optimization is Q -2 and converges to x^* from two sides alternately.

Key words　one-dimensional optimization; conic method; convergence rate

考虑 1 维无约束极小化问题

$$\min_{x\in \mathbf{R}} f(x) \tag{1}$$

其中 $f(x)$是 **R** 上的非线性光滑函数，假设当前迭代点为 x_k，为了得到下一个迭代点 x_{k+1}，在 x_k 附近，采用锥模型方法时由下面的锥函数来近似目标函数 $f(x)$：

$$m_k(x)=a_k+b_k\frac{x-x_k}{1+d_k(x-x_k)}+\frac{1}{2}c_k\left(\frac{x-x_k}{1+d_k(x-x_k)}\right)^2$$

其中 a_k, b_k, c_k, d_k 均为待定系数。以 $m_k(x)$的极小点作为 x_{k+1}。

由文献[1]知，如果 $c_k>0$ 且 $1+d_kc_k^{-1}b_k\neq 0$，则有

$$x_{k+1}=x_k-\frac{c_k^{-1}b_k}{1+d_kc_k^{-1}b_k} \tag{2}$$

为了确定 a_k, b_k, c_k, d_k，要求 $m_k(x)$满足下列插值条件：$m_k(x_k)=f(x_k)$，$m_k(x_{k-1})=f(x_{k-1})$，$m_k'(x_k)=g(x_k)=f'(x_k)$，$m_k'(x_{k-1})=g(x_{k-1})=f'(x_{k-1})$，可以得到

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$$a_k = f(x_k)$$
$$b_k = g(x_k)$$
$$c_k = \frac{g(x_k)\beta_k - g(x_{k-1})\beta_k^3}{p_{k-1}}$$
$$d_k = \frac{1-\beta_k}{p_{k-1}}$$

其中:$p_{k-1} = x_k - x_{k-1}$, $\beta_k = \dfrac{f(x_{k-1}) - f(x_k) + \rho_k}{-g(x_{k-1})p_{k-1}}$, $\rho_k^2 = [f(x_k) - f(x_{k-1})]^2 - g(x_k)g(x_{k-1})p_{k-1}^2$, $\rho_k = (\rho_k^2)^{1/2}$;从而由式(2)可以得到

$$x_{k+1} = x_k - \frac{g(x_k)p_{k-1}}{g(x_k) - g(x_{k-1})\beta_k^3} \tag{3}$$

若式(3)中 $\beta_k \equiv 1$,则此时锥模型方法退化为割线法。

文献[2]证明了锥模型方法是超线性收敛的,即

$$\lim_{k\to\infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|} = 0$$

其中 x^* 为问题(1)的解,从而有

$$\lim_{k\to\infty} \frac{|x_k - x_{k-1}|}{|x_{k-1} - x^*|} = 1$$

下面进一步讨论对 1 维优化问题锥模型算法的收敛阶。利用泰勒展开,注意到 $f(x^*) = 0$ 及 $\delta_k = \max\{|x_k - x^*|, |x_{k-1} - x^*|\}$,有

$$f(x_k) - f(x_{k-1}) = \frac{1}{2}f''(x^*)p_{k-1}(x_k + x_{k-1} - 2x^*) + \frac{1}{6}f'''(x^*)p_{k-1}[(x_k - x^*)^2 + (x_k - x^*)(x_{k-1} - x^*) + (x_{k-1} - x^*)^2] + O(\delta_k^4)$$

$$g(x_k)g(x_{k-1}) = f''^2(x^*)(x_k - x^*)(x_{k-1} - x^*) + \frac{1}{2}f''(x^*)f'''(x^*)(x_k - x^*)\cdot (x_{k-1} - x^*)(x_k + x_{k-1} - 2x^*) + O(\delta_k^4)$$

从而

$$\rho_k^2 = \frac{1}{4}f''^2(x^*)p_{k-1}^4\left[1 + \frac{2}{3}\frac{f'''(x^*)}{f''(x^*)}(x_k + x_{k-1} - 2x^*)\right] + O(\delta_k^6)$$

$$p_k = \frac{1}{2}f''(x^*)p_{k-1}^2\left[1 + \frac{1}{3}\frac{f'''(x^*)}{f''(x^*)}(x_k + x_{k-1} - 2x^*)\right] + O(\delta_k^4)$$

又可知

$$\beta_k = \frac{f(x_k) - f(x_{k-1}) - \rho_k}{g(x_{k-1})p_{k-1}} = 1 + \frac{1}{6}\frac{f'''(x^*)}{f''(x^*)}p_{k-1} + O(\delta_k^3)$$

$$\beta_k^3 = 1 + \frac{1}{2}\frac{f'''(x^*)}{f''(x^*)}p_{k-1} + O(\delta_k^2) \tag{4}$$

由式(3)和(4)知

$$x_{k+1} - x^* = x_k - x^* - \frac{g(x_k)p_{k-1}}{g(x_k) - g(x_{k-1})\beta_k^3} =$$

$$[g(x_k) - g(x_{k-1})\beta_k^3]^{-1}\{[g(x_k) - g(x_{k-1})\beta_k^3](x_k - x^*) - g(x_k)p_{k-1}\} =$$

$$\cdot\left\{f''(x^*)p_{k-1}\left[1+\frac{1}{2}\frac{f'''(x^*)}{f''(x^*)}(x_k-x^*)+O(\delta_k^2)\right]\right\}^{-1}\left\{-\frac{1}{4}\frac{f'''^2(x^*)}{f''(x^*)}p_{k-1}(x_{k-1}-x^*)^2\cdot\right.$$

$$\left.(x_k-x^*)-\frac{1}{6}f^{(4)}(x^*)p_{k-1}(x_k-x^*)^3+O(\delta_k^5)\right\}=\frac{1}{f''(x^*)}\left[1-\frac{1}{2}\frac{f'''(x^*)}{f''(x^*)}(x_k-x^*)+O(\delta_k^2)\right]\cdot$$

$$\left[-\frac{1}{4}\frac{f'''^2(x^*)}{f''(x^*)}(x_{k-1}-x^*)^2(x_k-x^*)-\frac{1}{6}f^{(4)}(x^*)(x_k-x^*)^3+O(\delta_k^4)\right]=$$

$$-\frac{1}{4}\frac{f'''^2(x^*)}{f''^2(x^*)}(x_{k-1}-x^*)^2(x_k-x^*)-\frac{1}{6}\frac{f^{(4)}(x^*)}{f''(x^*)}(x_k-x^*)^3+O(\delta_k^4)=$$

$$-\frac{1}{4}\frac{f'''^2(x^*)}{f''^2(x^*)}(x_{k-1}-x^*)^2(x_k-x^*)+o(|x_{k-1}-x^*|^2|x_k-x^*|) \tag{5}$$

所以

$$\lim_{k\to\infty}\frac{|x_{k+1}-x^*|}{|x_{k-1}-x^*|^2|x_k-x^*|}=\frac{1}{4}\frac{f'''^2(x^*)}{f''^2(x^*)}\triangleq a$$

令 $e_k=|x_k-x^*|$，$\varepsilon_k=a^{1/2}e_k>0$，则当 k 充分大时，由式(5)可知

$$\varepsilon_{k+1}=\varepsilon_k\varepsilon_{k-1}^2 \tag{6}$$

令 $\eta_k=\varepsilon_{k+1}/\varepsilon_k^2$，则式(6)变为 $\eta_k=(\eta_{k-1})^{-1}$，利用归纳法可知

$$\lim_{k\to\infty}\eta_{2k}=\frac{1}{\eta_0},\ \lim_{k\to\infty}\eta_{2k+1}=\eta_0$$

由此可知$\{\eta_k\}$是一有界序列，所以存在 $M>0$，使 $\eta_k=\varepsilon_{k+1}\varepsilon_k^{-2}\leqslant M$，即 $\varepsilon_{k+1}\leqslant M\varepsilon_k^2$，所以

$$|x_{k+1}-x^*|\leqslant M|x_k-x^*|^2$$

这表明$\{x_k\}$是Q-2阶收敛于 x^* 的；从式(5)又可知，$\{x_k\}$是从 x^* 的两侧交替收敛到 x^* 的。由以上的讨论，可以得到

定理 假设 $f(x)\in C^4$，x^*是问题(1)的解且 $f'''(x^*)\neq 0$，则存在 $\delta>0$，使得当 $|x_0-x^*|<\delta$ 时，由锥模型方法的迭代公式(3)产生的点列$\{x_k\}$Q-2阶收敛于 x^*，即存在 $M>0$，使

$$|x_{k+1}-x^*|\leqslant M|x_k-x^*|^2$$

且$\{x_k\}$点列从 x^* 的两侧收敛于 x^*。

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